Reliability and Risk Analysis

Markov Models for Risk Assessment
We consider a discrete random process (a random sequence) $X(t), \ t \in T = \{0, 1, 2, \ldots \}$ with a finite number of states $E = \{1, 2, \ldots, N\}$. In case that $X(t) = i$, we say that the process is at time $t$ in the state $i$, $t \in T$, $i \in E$. The states of $X(t)$ can be different risk situations.
**Markov property** – the future development of the process at time $t + 1$ depends only on state of the process at time $t$, and does not depend on past development at times $t - 1$, $t - 2$, ..., $2, 1, 0$. We can describe is as follows: for all $t = 0, 1, 2, \ldots$ and all states $i, j, i_{t-1}, \ldots, i_0 \in E$ is

$$P(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0) = P(X_{t+1} = j | X_t = i).$$

(1)

Such a process is called the **Markov chain**.
We introduce notation

\[ p_{ij}(t, t + 1) = P(X_{t+1} = j | X_t = i) \]

Probabilities \( p_{ij}(t, t + 1) \) are called the \textbf{transition probabilities from the state} \( i \) \textbf{at time} \( t \) \textbf{to state} \( j \) \textbf{at time} \( t + 1 \).

Probabilities

\[ p_{ij}(t, t + s) = P(X_{t+s} = j | X_t = i), \]

are called \textbf{transition probabilities from the state} \( i \) \textbf{at time} \( t \) \textbf{to state} \( j \) \textbf{at time} \( t + s \).
I some cases, transition probabilities $p_{ij}(t, t + s)$ do not depend on $t$ and $t + s$, but only on their difference $s$. We call such chains \textbf{homogeneous}. We will focus only on homogeneous chains. We will use notation:

\begin{align*}
    p_i(0) &= P(X(0) = i) & \text{initial probability of state } i, \\
    \mathbf{p}(0) &= (p_1(0), p_2(0), \ldots, p_N(0)) & \text{initial distribution of Markov chain states,} \\
    p_i(t) &= P(X(t) = i) & \text{absolute probability of state } i \text{ at time } t, \\
    \mathbf{p}(t) &= (p_1(t), p_2(t), \ldots, p_N(t)) & \text{absolute distribution of Markov chain states at time } t.
\end{align*}
Instead of \( p_{ij}(t, t + 1) \), we can for homogeneous Markov chain write \( p_{ij}(1) = p_{ij} \). The probability \( p_{ij} \) we call transition probability after one step and the matrix

\[
P = \begin{pmatrix}
p_{11}, p_{12}, \ldots, p_{1N} \\
p_{21}, p_{22}, \ldots, p_{2N} \\
\vdots \\
p_{N1}, p_{N2}, \ldots, p_{NN}
\end{pmatrix}
\]

we call the **transition probability matrix of the homogeneous Markov chain** or briefly **transition probability matrix**

For each row of \( P \) is \( \sum_{j=1}^{N} p_{ij} = 1 \), the matrix \( P \) is a stochastic matrix.
We can describe dynamics of the process $X(t)$ using the matrix $P$ and the vector of initial states $p(0)$

$$P(X(0) = i_0, X(1) = i_1, \ldots, X(k) = i_k) = p_{i_0}(0)p_{i_0i_1}p_{i_1i_2} \cdots p_{i_{k-1}i_k}$$

for any states $i_0, i_1, \ldots, i_k \in E$. We denote

$$p_{ij}^{(s)} = p_{ij}(t, t + s) \text{ for } s = 1, 2, \ldots \quad \text{a} \quad p_{ij}^{(0)} = \begin{cases} 
0 & \text{for } i \neq j, \\
1 & \text{for } i = j.
\end{cases}$$
The probability $p^{(s)}_{ij}$ is the transition probability from the state $i$ to the state $j$ after $s$ steps. We can write them into the matrix

$$
P^{(s)} = \begin{pmatrix} p^{(s)}_{11}, \ldots, p^{(s)}_{1N} \\ \vdots \\ p^{(s)}_{N1}, \ldots, p^{(s)}_{NN} \end{pmatrix},$$

which is called the transition probability matrix of the homogeneous Markov chain after $s$ steps.
Markov chain

\[ p_{ij}^{(2)} = P(X(t + 2) = j | X(t) = i) = \]
\[ = \sum_{k=1}^{N} P(X(t+2) = j | X(t+1) = k, X(t) = i) \cdot P(X(t+1) = k | X(t) = i) = \]
\[ = \sum_{k=1}^{N} P(X(t+2) = j | X(t+1) = k) P(X(t+1) = k | X(t) = i) = \sum_{k=1}^{N} p_{kj} p_{ik} \]

for \( i, j \in E \). In the matrix form we have

\[ P^{(2)} = P^2. \]

By analogy

\[ P^{(s)} = P^s. \]
Markov chain

\[ p_i(t) = P(X(t) = i) = \sum_{k=1}^{N} P(X(0) = k)P(X(t) = i|X(0) = k) = \]

\[ = \sum_{k=1}^{N} p_k(0)p_{ki}^{(t)}, \quad i \in E. \]

In the matrix form we have

\[ p(t) = p(0)P^{(t)} = p(0)P^t. \quad (2) \]
In terms of long-term of the chain $X(t)$ it is useful to determine absolute probability of states $p_i(t)$ for large $t$ ($t \to \infty$), we want to find $\lim_{t \to \infty} p_i(t)$, $i \in E$.

$$\lim_{t \to \infty} p_{ik}^{(t)} = \pi_k,$$
$$\lim_{t \to \infty} p_k(t) = \pi_k$$

for $i, k \in E$, where $\pi_1, \pi_2, \ldots, \pi_N$ are unique solutions of

$$\pi_k = \sum_{j=1}^{N} \pi_j p_{jk}, \quad k \in E \quad \text{and} \quad \sum_{j=1}^{N} \pi_j = 1.$$  \hspace{1cm} (3)
We denote \( \pi = (\pi_1, \ldots, \pi_N) \) and

\[
\Pi = \begin{pmatrix}
    \pi \\
    \pi \\
    \vdots \\
    \pi
\end{pmatrix} = \begin{pmatrix}
    \pi_1, \pi_2, \ldots, \pi_N \\
    \pi_1, \pi_2, \ldots, \pi_N \\
    \vdots \\
    \pi_1, \pi_2, \ldots, \pi_N
\end{pmatrix},
\]

we can rewrite given limits in the matrix form as follows

\[
\lim_{t \to \infty} P^{(t)} = \lim_{t \to \infty} P^t = \Pi
\]

\[
\lim_{t \to \infty} p(t) = \lim_{t \to \infty} p(0)P^t = \pi,
\]

where \( \pi \) is a unique solution of

\[
\pi = \pi P, \quad \pi 1 = 1.
\]
The vector $\pi = (\pi_1, \ldots, \pi_N)$ defines so called the stationary probability distribution of the chain $X(t)$. If the initial probability distribution is stationary, ie $p(0) = \pi$, then all absolute probability distributions $p(t)$ are stationary, ie $p(t) = \pi$ and we say that the chain is in the statistic equilibrium.
Example – accident insurance model

The insurance company uses three categories of insurance: 0 – basic, 1 – bonus 30% and 2 – bonus 50%. Let $X(t)$ be a random process which indicates the insurance category in the period $t$, $E = \{0, 1, 2\}$. The insured person is added to the appropriate category depending on the number of insurance events reported in the previous period. The insured person is in the first period of insurance categorized as 0 – basic insurance. If the insurance policyholder claims-free record in the first period, he/she is included in category above (gets bonus) in the subsequent period. However, if one applies the insurance claim in the next period, he/she is classified in the lower category, applying more than one insurance claim, he/she is classified in the lowest category. Suppose that the number of claims in the insurance period $t$ is a random variable $Y_t$, $t = 1, 2, \ldots$. 
Example – accident insurance model

\[
X(t + 1) = \begin{cases} 
\min\{X(t) + 1; 2\} & \text{for } Y_t = 0, \\
\max\{X(t) - 1; 0\} & \text{for } Y_t = 1, \\
0 & \text{for } Y_t > 1. 
\end{cases}
\]

We assume that \( Y_t \) pro \( t = 1, 2 \ldots \) are independent random variables with the same Poisson distribution with the \( \lambda \). Then \( X(t) \) is the homogeneous Markov chain, \( E = \{0, 1, 2\} \), with the initial probability distribution \( p(0) = (1, 0, 0) \) and the transition probability matrix

\[
P = \begin{pmatrix}
1 - e^{-\lambda} & e^{-\lambda} & 0 \\
1 - e^{-\lambda} & 0 & e^{-\lambda} \\
1 - e^{-\lambda} - \lambda e^{-\lambda} & \lambda e^{-\lambda} & e^{-\lambda}
\end{pmatrix}.
\]
Example – accident insurance model

Since the matrix $P$ has in the first column non-negative elements, there exists stationary distribution $\pi = \lim_{t \to \infty} p(t)$ and we obtain it by solving (3). If we set $a_0 = e^{-\lambda}$ and $a_1 = \lambda e^{-\lambda}$, we get

$$
\pi_0 = \pi_0 (1 - a_0) + \pi_1 (1 - a_0) + \pi_2 (1 - a_0 - a_1),
$$

$$
\pi_1 = \pi_0 a_0 + \pi_2 a_1,
$$

$$
\pi_2 = \pi_1 a_0 + \pi_2 a_0,
$$

$$
\pi_0 + \pi_1 + \pi_2 = 1.
$$

$$
\pi_0 = \frac{1 - a_0 - a_0 a_1}{1 - a_0 a_1} = \frac{1 - e^{-\lambda} - \lambda e^{-2\lambda}}{1 - \lambda e^{-2\lambda}},
$$

$$
\pi_1 = \frac{a_0 (1 - a_0)}{1 - a_0 a_1} = \frac{e^{-\lambda} (1 - e^{-\lambda})}{1 - \lambda e^{-2\lambda}},
$$

$$
\pi_2 = \frac{a_0^2}{1 - a_0 a_1} = \frac{e^{-2\lambda}}{1 - \lambda e^{-2\lambda}}.
$$